<table>
<tr><td colspan="5">Mathematics Project Competition (2025/26)
數學專題習作比賽 (2025/26)
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<tr><td>Category
參賽組別</td><td colspan="4">✓ * A 組：初中習作 (Category A: Junior secondary project)
☐* B 組：中一小型習作 (Category B: S1 mini-project)</td></tr>
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專題習作題目</td><td colspan="4">Investigation on Alternative Solutions and Generalization of a HKMO problem</td></tr>
<tr><td>Name of School</td><td colspan="4">Kwun Tong Government Secondary School</td></tr>
<tr><td>學校名稱</td><td colspan="4">觀塘官立中學</td></tr>
<tr><td rowspan="2">Teacher supervisor
負責教師</td><td colspan="2">Name in English</td><td colspan="2">中文姓名</td></tr>
<tr><td colspan="2">Mr/~~Ms/Dr~~# Tsang Chun Shan</td><td colspan="2">曾震山 先生/~~女士/博士~~#</td></tr>
<tr><td rowspan="7">Team members
隊員</td><td colspan="2">Name in English</td><td>中文姓名</td><td>Class 班別</td></tr>
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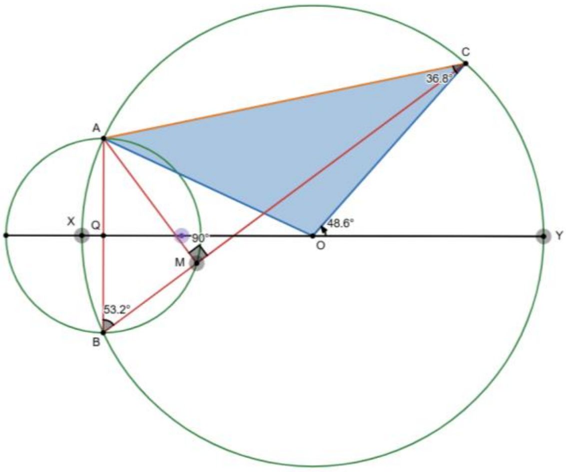
Investigation on Alternative Solutions and Generalization of a HKMO problem



Project Key Results

1. The length of AC is $R\sqrt{2}$ where R is the radius of larger circle with center at O
2. $ACOM$ is a cyclic quadrilateral with AC as the diameter
3. When the ratio XQ: QO decreases, $\triangle AOC$ keep constant but rotates clockwisely
4. When M move along arc AMB, $\angle ABM$ changes, so as length of AC
5. Rate of change of rotation of $\triangle AOC$ is given by:

$$m = Slope\ of\ OC = \frac{n}{\sqrt{2n+1}}$$

$$\frac{dm}{dt} = \frac{n+1}{(2n+1)^{\frac{3}{2}}} \cdot \frac{dn}{dt}$$

6. Rate of change of (curved surface area VAC in a cone = A) is given by:

$$\frac{dA}{dt} = \frac{\pi R^2}{360°}\left(\frac{d}{dt}\angle AOC\right)$$

Desmos Graph: https://www.desmos.com/geometry/eegwo2bjqo?lang=zh-TW

Table of Content

Part 1: Introduction

Motivation

Geometry, a branch of mathematics which concerns the properties and relations of points, lines, surfaces, solids, and higher dimensional analogues, is utilized in different fields frequently, such as architectural designing and astronomy. When people are trying to solve problems in such fields, they might use the fundamental formulas of geometry to solve them. However, it takes several times for people to solve problems since they have used methods that are not effective. Therefore, we are going to find what factors we should consider first when solving geometry problems so as to solve the same question from different approaches.

We have chosen a HKMO problem that related to geometry and we observe that this problem has many solutions, which benefits our researching on geometry. So we will try to find out different kinds of solutions and explore conjectures of this useful problem to search out the factors we should consider when solving geometry problems.

To solve this problem, we are trying to find out the relationships between different lines in a geometry diagram and suggesting some conjectures that relate to the diagram.

Eventually, we would use the relationship of lines in the diagram to conduct a formula which helps to solve the geometry question. Afterwards, we would do deeper exploration for this formula to see if there are any applications in other geometry problems for it.

The original HKMO problem we use in this math project: [2023 - 2024 Heats Individual Q14]

In Figure 4, XY is a diameter of the circle with centre at O and radius 5 cm. XY intersects the chord AB at Q such that $\angle AQO = 90°$ and $XQ = QO$. A semi-circle with diameter AB intersects XY at M。 BM produced intersects the circle at C. Find the length of AC.

圖四中，XY 是一個以 O 為圓心及半徑為 5 cm 的圓的直徑。XY 與弦 AB 相交於點 Q，使得 $\angle AQO = 90°$ 及 $XQ = QO$。以 AB 為直徑的半圓與 XY 相交於 M，延線 BM 與圓相交於點 C，求 AC 的長。

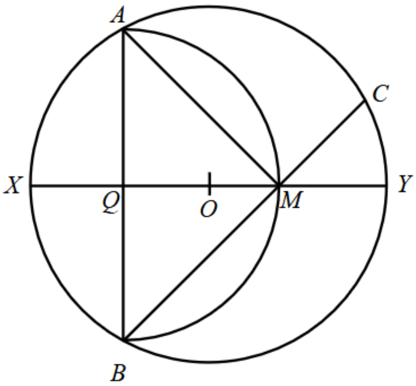


Figure 4

圖四

Problem Statement

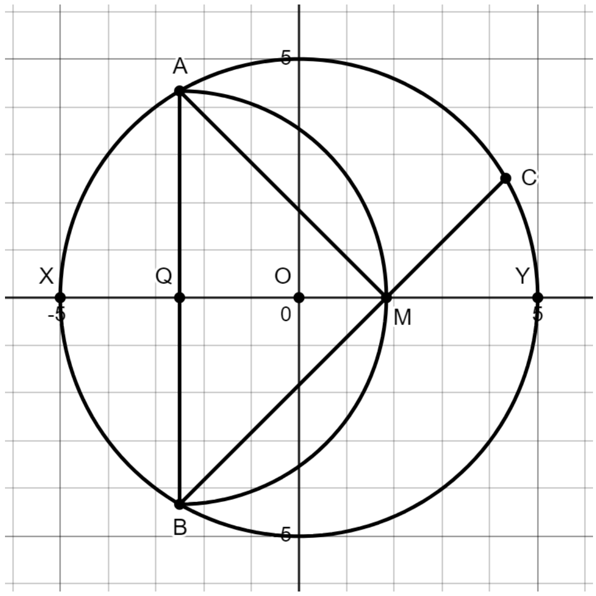


Figure 1

In Figure 1, XY is the diameter of the circle with centre at O and radius 5 cm. XY intersects the chord AB at Q such that $\angle AQO=90°$ and $XQ = QO$. A semi-circle with diameter AB intersects XY at M. BM is produced to intersect the circle at C. We introduce a coordinate system in the original problem with $O(0,0)$ as the centre of the circle with diameter XY and radius 5 units.

Questions:
What is the length of AC? Are there any the alternative solutions to find AC?
Are there any special triangles? Are there any cyclic quadrilaterals?
What if the ratio XQ:QX is generalized from 1:1 to 1:n, what will be the effects?
What if $\angle ABM$ changes along with the change in the ratio XQ:QX, how to describe the effect?

Notation

In the project, we can form different circles, quadrilaterals and triangles. We make notations on the figures we will encounter in this project.

Denote C_1 as the circle with centre at point O and radius = OX = OY = 5 cm
Denote C_2 as the circle with centre at point Q and radius = QA = QB = QM = 2.5 cm

Part 5: Conclusion and Further Work

Conclusion:

We have conducted three different ways to calculate the length of AC in original problem. Furthermore, we have explored some conjectures and given their proofs and disproof. Some conjectures are insightful, for example, AC is the diameter of a cyclic quadrilateral which is not shown in the original problem.

We also propose five proposition that seek to generalize the original problem into a more general situation. The change in the ratio XQ:QO from 1:1 to 1: n is our major use for generalization. We also notice that $\angle ABC$ is another important variable that can be used for generalization. In the process of generalization, we have deduced the following interesting formula.

$$Slope\ of\ OC = \frac{n}{\sqrt{2n+1}}$$

$$\frac{dm}{dt} = \frac{n+1}{(2n+1)^{\frac{3}{2}}} \cdot \frac{dn}{dt}$$

$$\frac{dA}{dt} = \frac{\pi R^2}{360°}\left(\frac{d}{dt}\angle AOC\right)$$

Particularly, we are amazed to find in the first formula which states that the slope of OC is a function of n in the ratio XQ:QO = 1:n !

Further Work:

This project has many interesting characteristics to explore and to generalize. Here are some of the ideas that we would like to explore further:

- Using function f(x) to describe the rotation of triangle
- Extend from 2D plane geometry to 3D conic geometry
- Proposed generalized formula from this problem in a 3D setting

All in all, this math project inspires us to learn the math knowledge and skills such as law of cosines and quadratic formula that are needed to solve this problem. In addition, we have developed our mathematical thinking mindset such as conjecturing, proving and writing logical arguments.

Part 6: Student’s Reflection

Reflection of Student 1

When I first look at this question, I think this is too hard for me but at the end of the project, I realize that this is very fun to study, and I learn that one question can use a lot of method to solve it. I also learn how to find coordinates of a point and how to find slope of a line. I hope I can join some more project in the future.

Reflection of Student 2

In this project, I have learnt a lot besides mathematics knowledges and get satisfaction during the investigation. When I solved a question successfully, I feel satisfied. It feels like you destroy the obstacles that block your road to success. Yet, I have encountered some challenges in problem solving, but my teammates can help me to solve them through discussion. We discuss the problems together and finally got the solution for them. Therefore, I realized that doing co-operation gives impressive power in problem solving, it helps you to solve hard questions with acceptable velocity. Also, it strengthens my communication skills through discussing with my teammates during the investigation. As a result, I would say this project is wonderful. It has boosted my mathematical skills and strengthen my communication skills in co-operations, which is useful. I believe that these improvements have laid the foundations for me to deal with more investigations in the future!

Reflection of Student 3

In this project, I have learnt a lot of knowledge about mathematics. It can improve my logical thinking and problem-solving skills.

References

https://www.desmos.com/geometry/eegwo2bjqo?lang=zh-TW
https://www.desmos.com/geometry/nmdox1vwgy?lang=zh-TW