Mathematics Project Competition (2025/26)
數學專題習作比賽 (2025/26)
Information Sheet 資料頁

Category 參賽組別	✔ A 組：初中習作 (Category A: Junior secondary project) ☐* B 組：中一小型習作 (Category B: S1 mini-project)			
Title of Project 專題習作題目	**Radii of Inscribed Circles in Regular Polygons**			
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* Please "✔" whichever is appropriate. 請在適用部分加上"✔"
\# Please delete whichever is inappropriate. 請刪去不適用部分

Radii of Inscribed Circles in Regular Polygons

Pui Ching Middle School

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1. Introduction

In a triangle, we can construct an inscribed circle (or in-circle). (See Figure 1) The radius of the inscribed circle is determined by the side lengths of the triangle. In this project, we are interested in finding the radii of inscribed circles of triangles formed by points of a polygon.

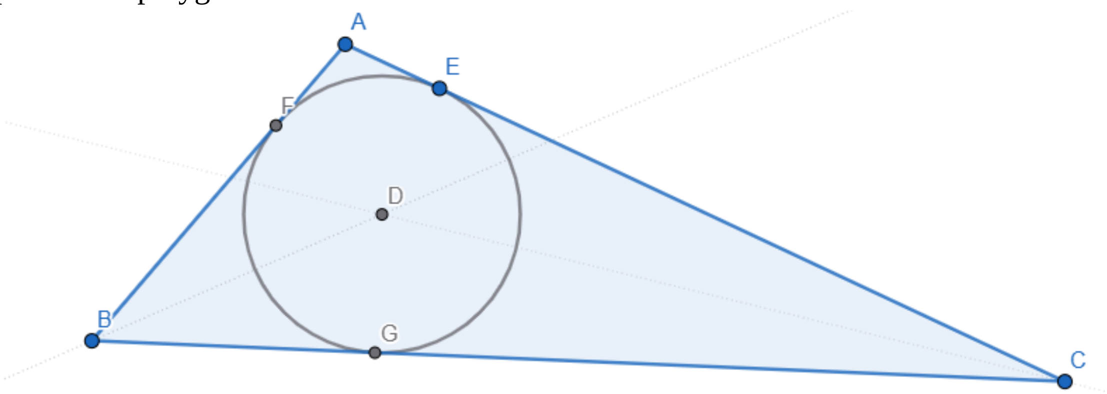


Figure 1 (Triangle with a in-circle)

We will consider some easier case first, but not all polygons. So we will only consider polygons that is a regular polygon with side length 1.

Firstly, we should define points in a regular polygon.

Definition 1.1.

Let $n \geq 3$. Denote the vertices of a regular n-sided polygon with side length 1 by A_1, A_2, …, A_n in anti-clockwise direction. (See Figure 2).

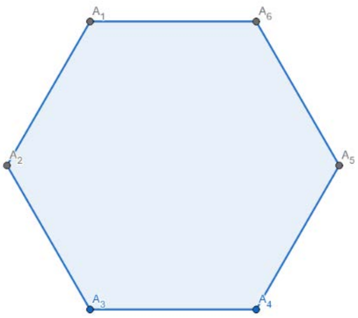
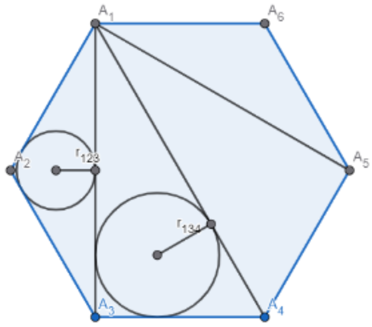


Figure 2

Choosing any three points form a triangle and determine an inscribed circle. We attempt to find the radii of these inscribed circles. The following definition gives the name of the variables.

Definition 1.2.

Let $n \geq 3$ and $1 \leq i < j < k \leq n$. Denote r_{ijk} as the radius of the inscribed circle of the triangle $A_iA_jA_k$.



Since r_{ijk} is determined by the side lengths A_iA_j, A_jA_k and A_iA_k, we need to find their lengths. The following definition is based on the symmetry that the length of A_iA_j is determined by the difference between i and j.

Definition 1.3.

Let $n \geq 3$ and $1 \leq i \leq j \leq n$. Let $d = j - i$. Denote l_d as the length of A_iA_j.

Notice that $l_0 = 0$, $l_1 = 1$ and $l_i = l_{n-i}$.

The following result tells us the relation between the radius of the inscribed circle and the side lengths of the triangle.

Lemma 1.4.

Let a, b, and c be the side lengths of a triangle and r is the radius of the inscribed circle.

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \text{ where } s = \frac{a+b+c}{2}.$$

Proof. Since $A = rs$, we have $r = \frac{A}{s}$. By the *Heron's Formula*, we have

$$\begin{aligned} r &= \frac{A}{s} \\ &= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} \\ &= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2}} \\ &= \sqrt{\frac{(s-a)(s-b)(s-c)}{}} \end{aligned}$$

3. Conclusion and Analysis

Now, the cutting line can be found by only one simple formula. Therefore, we can substitute the previous theorem.

Theorem 3.1. The general term of r_{ijk} is

$$r_{ijk} = \frac{2}{\sin\frac{\pi}{n}} \sin\frac{(j-i)\pi}{2n} \sin\frac{(k-j)\pi}{2n} \cos\frac{(k-i)\pi}{2n}$$

Proof.

We find $r_{i,j,k}$ by *Lemma 1.4*,
$a = l_{j-i} = 1$, $b = l_{k-j}$, $c = l_{k-i}$.

into the *Heron's Formula*, then by sum-to-product, we obtain the desired result.