Mathematics Project Competition (2025/26)
數學專題習作比賽 (2025/26)
Information Sheet 資料頁

Category 參賽組別	✓* A 組：初中習作 (Category A: Junior secondary project) ☐* B 組：中一小型習作 (Category B: S1 mini-project)			
Title of Project 專題習作題目	Minimum Area of Triangle Enclosed by a Rubber Band on a “n×n” Geoboard			
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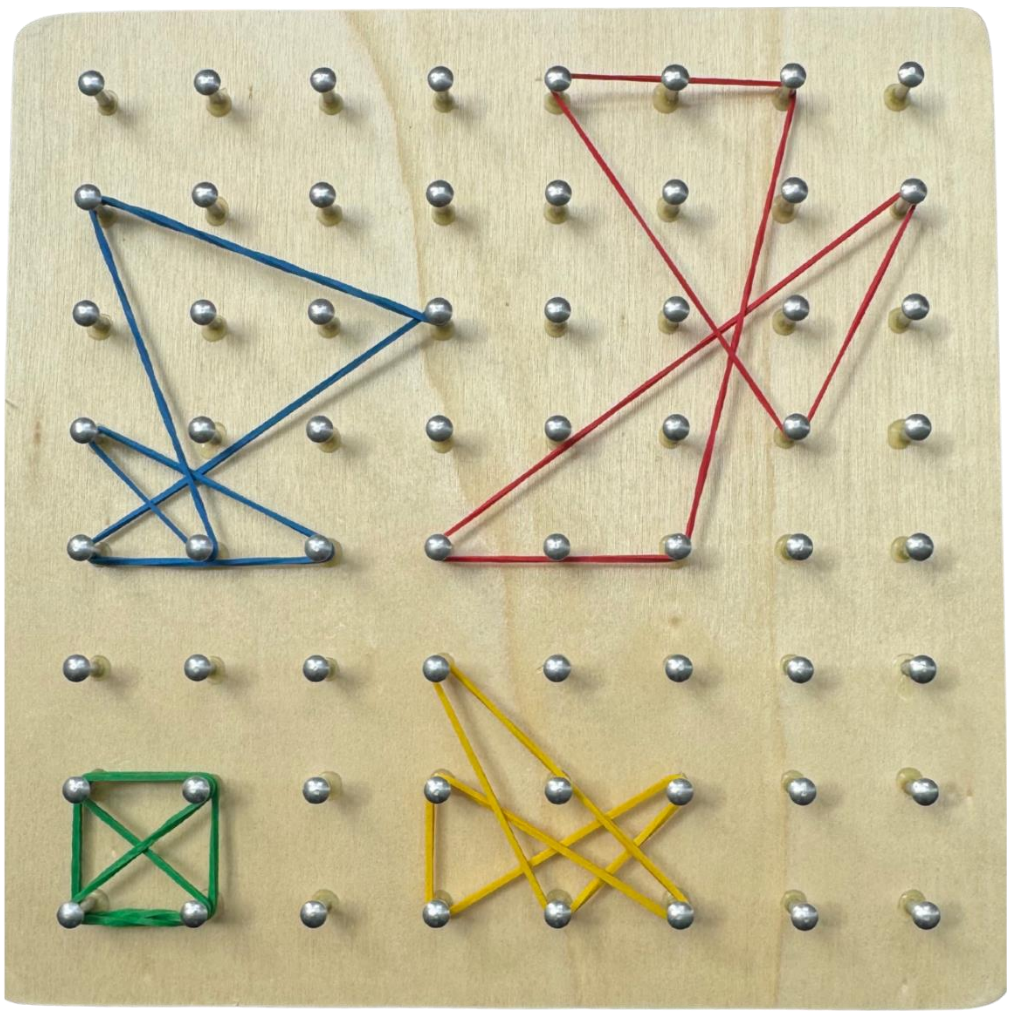
	4	YIP TSZ HIM	葉梓謙	3D
	5			
	6			

* Please “✓” whichever is appropriate. 請在適用部分加上“✓”

\# Please delete whichever is inappropriate. 請刪去不適用部分

Minimum Area of Triangle Enclosed by a Rubber Band on a “$n \times n$” Geoboard



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1. Abstract

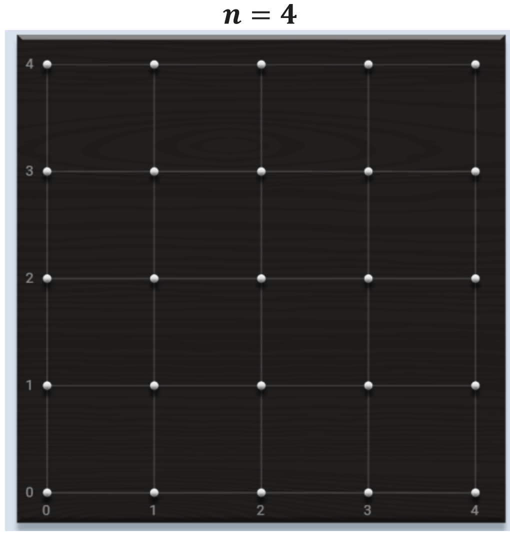
Beyond the traditional constraints of Pick's Theorem, allowing a rubber band to twist and intersect itself on a geoboard forms triangles with areas smaller than $\frac{1}{2}$ square units. In this project, we are investigating the minimal enclosed areas by three straight lines on an "$n \times n$" geoboard for various values of n in this project and found a formula $A_{\min}(n) = \frac{1}{4(n-1)^2[n^2(n-1)^2-1]}$ for the cases even $n \geq 4$.

2. Introduction

The Classic $n \times n$ Geoboard

We define the $n \times n$ geoboard as a discrete lattice consisting of $(n+1)^2$ integer lattice points (x, y) where both x and y are integers from 0 to n.

$\boldsymbol{n = 4}$



Definition: We define the bottom-left lattice of the geoboard as $(0,0)$ and the top-right lattice as (n, n) on an $n \times n$ geoboard.

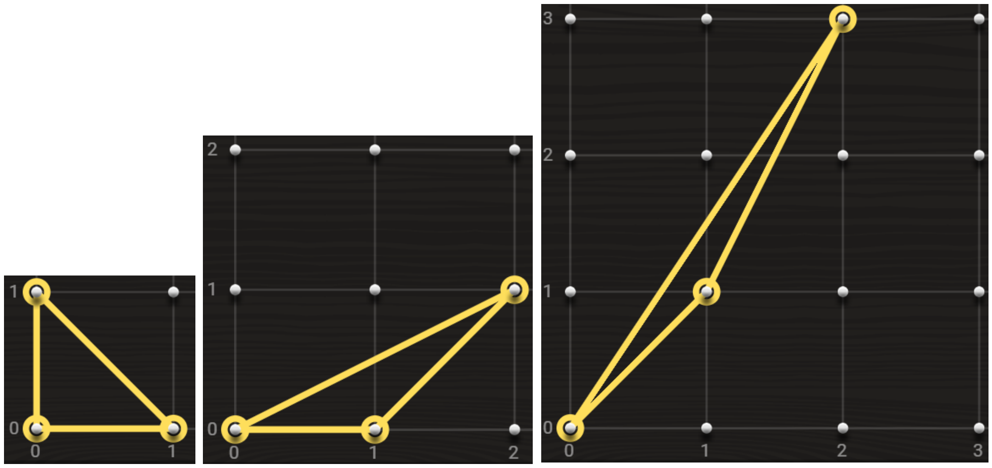
Triangles on Integral Vertices (Pick's Theorem)

For simple polygons with vertices at lattice points, Pick's Theorem states:

$$A = I + \frac{B}{2} - 1$$

where I = interior lattice points and B = boundary lattice points.

The following figures show some examples of such polygons:



For any triangle with 3 vertices on lattice points without extra boundary points and no interior points:

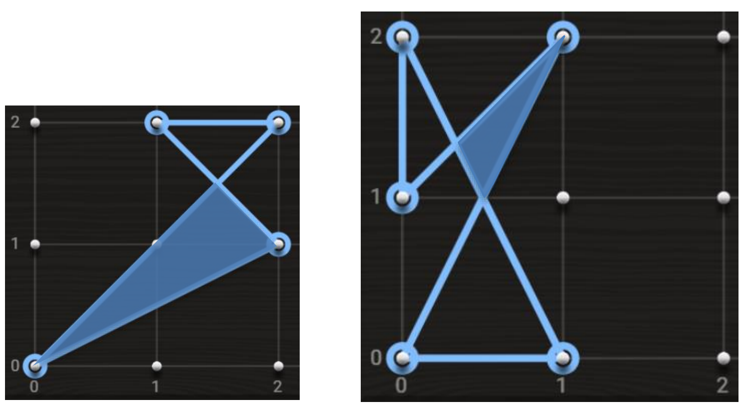
$$A = 0 + \frac{3}{2} - 1 = \frac{1}{2} \text{ square unit.}$$

Thus, the minimum area under Pick's Theorem is $\boxed{\frac{1}{2}}$ square unit.

Self-Intersecting Rubber Bands (New Approach)

For this project, we remove the non-self-intersecting constraint. The rubber band can "flip" or intersect itself, creating multiple enclosed regions where vertices of the triangles are intersection points of lines, not necessarily lattice points.

The following figures show some examples of such polygons:



New Question: What is the minimal enclosed area if we allow the rubber band to self-intersect and vertices can be at non-lattice intersection points?

3. Methodology

3.1 Problem Formulation

Instead of considering the whole rubber band, we actually seek three straight lines, each passing through at least 2 lattice points on the $n \times n$ geoboard, such that the enclosed triangular area is minimized.

3.2 Rubber Band Assumptions

- The rubber band is infinitely thin (consistent with Euclid's definition of lines)
- It passes through integer lattice points but can cross itself
- It is always stretched tight with no slack

3.3 Determinant Formula for Triangle Area

Given three lines in standard form:

$$L_1: a_1x + b_1y + c_1 = 0$$

$$L_2: a_2x + b_2y + c_2 = 0$$

$$L_3: a_3x + b_3y + c_3 = 0$$

The area of the triangle formed by their intersection is:

$$\boxed{\text{Area} = \frac{|\Delta|^2}{2M}}$$

where:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and

$$M = |D_{12} \cdot D_{23} \cdot D_{31}|$$

with $D_{ij} = a_i b_j - a_j b_i$ (pairwise determinants).

8.3 Main Conclusions

Conclusion 1: Allowing self-intersection breaks the $\frac{1}{2}$ barrier

Pick's Theorem implies that for simple lattice triangles, the smallest possible area is $\frac{1}{2}$. In this project we removed the non-self-intersection constraint by allowing the "rubber band" to cross itself, so the triangle vertices can be intersection points of lines rather than lattice points.

This allows areas far smaller than $\frac{1}{2}$, and in the even-n construction the area decreases roughly on the order of n^{-6}.

Conclusion 2: The unimodular condition and minimal-area formula for even $n \geq 4$

In all best configurations we found, the determinant Δ satisfies $|\Delta| = 1$. This supports the principle:

To minimize the area $\frac{|\Delta|^2}{2M}$, we must first force $|\Delta|$ to be as small as possible, i.e. $|\Delta| = 1$, and then maximize M. For even $n \geq 4$, our construction achieves this and leads to the explicit area formula

$$A_{\min}(n) = \frac{1}{4(n-1)^2[n^2(n-1)^2 - 1]}$$

which we have rigorously proved for our family of lines and shown to be locally optimal among unimodular triples.

Conclusion 3: Parity (even vs odd n) changes the geometry of the problem

A major theme of this investigation is the stark structural difference between even and odd board sizes:

Even $\boldsymbol{n}$: The board has a true central lattice row at $\boldsymbol{y} = \frac{\boldsymbol{x}}{\mathbf{2}}$ (an integer coordinate), allowing a perfectly symmetric construction with a uniform closed-form area formula for all even $\boldsymbol{n} \geq \mathbf{4}$.

Odd $\boldsymbol{n}$: There is no such central row; the midpoint lies strictly between lattice rows. This breaks the symmetric structure and forces different arithmetic compromises for each odd value of $\boldsymbol{n}$. The best configurations are case-specific and less uniform.

Conclusion 4: Symmetry Explains Multiplicity

Every $n \times n$ geoboard has 8 fundamental symmetries (4 rotations and 4 reflections). Therefore each configuration of lines typically generates up to 7 additional equivalent configurations with identical area. Recognizing this symmetry is crucial for interpreting search output correctly and for avoiding double-counting in tables.

Conclusion 5: Farey Sequences Govern Optimality

The Farey sequence of order n —which lists all reduced fractions with denominator $\leq n$ —naturally governs the space of valid line slopes on the $n \times n$ geoboard (May not work in some odd case of n). Our optimal configuration uses the extreme Farey neighbours of 1 (the fractions closest to unity under the denominator bound) is a beautiful connection between lattice geometry and number theory.

8.4 Limitations

We proved the even-n area formula for our construction, but we did not prove global minimality for all even n.

For odd n, all results are empirical best candidates; minimality is unproved.

Exhaustive search becomes computationally expensive as n increases, even after using symmetry reductions.

8.5 Overall Summary

This project demonstrates that allowing self-intersecting rubber bands on geoboards leads to a rich mathematical landscape governed by determinants, Farey sequences, and unimodular transformations. For even boards, we have constructed an explicit family with a closed-form area formula and proved local and empirical global optimality. For odd boards, we have identified structural barriers (the absent central row) that prevent a uniform solution. The proofs in Section 5.3 (Lemmas 1–3) establish rigorous constraints on the space of possible configurations, providing strong evidence for global optimality despite leaving a complete proof as an open problem for future work.

9. References and Appendices

References:

[1] Pick's theorem: https://en.wikipedia.org/wiki/Pick%27s_theorem

[2] Determinant Formula for Triangle Area: Ali, S. T. (2003). "The Area of a Triangle Formed by Three Lines." The Mathematical Gazette, 87(510), 296–298. https://www.jstor.org/stable/24336991

[3] Farey sequence:
https://dummit.cos.northeastern.edu/teaching_sp21_4527/4527_lecture_03_farey_sequences_v2.pdf

[4] Symmetry of a square:

https://en.wikipedia.org/wiki/Dihedral_group_of_order_8

Tools used:

[1] Geoboard with rubber bands (shown on the cover page)

[2] Geogebra

[3] Virtual Geoboard https://apps.mathlearningcenter.org/geoboard/

[4] Perplexity Pro for
(i) Provide visualized possible outcomes for triangles formed on 3×3 geoboard with determinant $= \pm 1$.
(ii) Exhaustive search in verifying the conjecture in using the formula to find the minimal enclosed area are true for $n = 4$ and $n = 6$,
(iii) Given the condition, finding the minimal area for odd n from 7 to 25,
(iv) Grammar Check and Correct Style of Writing.
(v) Checking of algebraic correctness of the formula