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負責教師</td><td colspan="2">Name in English</td><td colspan="2">中文姓名</td></tr>
<tr><td colspan="2">Mr/~~Ms/Dr~~# CHEUNG PAK LEONG</td><td colspan="2">張伯亮 先生/~~女士/博士~~#</td></tr>
<tr><td rowspan="7">Team members
隊員</td><td colspan="2">Name in English</td><td>中文姓名</td><td>Class 班別</td></tr>
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On Minimal Enclosing Regular Polygons of Regular Polygons

St. Paul's Co-educational College
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Abstract

Given any regular k-gon, what is the smallest regular n-gon that encloses it? This natural problem has been solved by Dilworth and Mane in "*On a problem of Croft on optimally nested regular polygons*" (2010), and our project has re-proved the following results by more elementary methods:

(i) $n = 3$

(ii) $n = 4$

(iii) Both n and k are even.

Our methodology may not be generalizable to tackle the other cases where at least one of n and k is odd.

Contents

1 Introduction

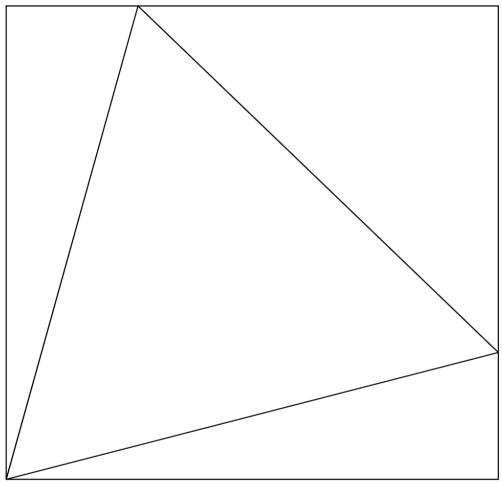

Figure 1. Common textbook problem

"*Figure 1 shows a diagram of the largest triangle that can fit inside the unit square. Find the side length of the triangle.*"

Problems like these are no mystery to secondary school students, for it is a good practice for pure geometric skills. Yet, why is that the biggest triangle that can fit inside a square placed like this? How can we take for granted? And what does the picture look like for other regular polygons? That is the question that we are looking to answer today.

During our journey to answer this question, we came across various articles related to regular enclosed polygons. In particular, "*Folding optimal polygons from squares*." by D. Dureisseix [2] used various interesting techniques to provide concrete methods for folding the largest regular k-gon with square origami paper. However, the article did not explicitly explain why the said configuration is indeed the largest regular k-gon. "*On a problem of Croft on optimally nested regular polygons*" by S.J. Dilworth & S.R. Mane [1] pointed out the relationship between the answer to this problem and the coprimality between of the number of sides of the polygons. We aim to re-prove part of their results by more elementary methods. "*Fits and covers*" by J. E. Wetzel [3] includes a wider range of results, way beyond just regular polygons, which motivated us to find other fixed shapes to enclose or be enclosed.

From the reasons above, we aim to prove the ratio between the two circumradii of minimally enclosing regular polygons. However, before diving into the geometric parts, it is necessary to do some set-up and preparation by proving some basic facts related to congruences, divisibility and functions and more, in Section 2.

In section 3, we aim to do the following: Given a regular k-gon with circumradius 1, find the minimal circumradius of an equilateral triangle enclosing that regular polygon. The first part will be to evaluate the lower bound of the circumradius, and the second will be to actually construct an equilateral triangle with the hypothesized lower bound.

In sections 4 and 5 we find the minimal enclosing strip and square for the regular k-gon. Section 4 provides us with a useful tool to apply in imposing the bound for section 6, and section 5 shows a simple application of the tool.

In section 6, we generalize our results to find the minimal enclosing regular $2n$-gon of a regular $2k$-gon, using both results in section 4 and number-theoric facts covered in section 2.

To conclude in section 7, we make basic observations on the minimal enclosing regular polygons to grasp a bigger picture on what these look like, to aid further exploration on the related topic. Moreover, we explain the potential difficulties one may face moving forward to generalizing our results even more.

7 Closing remark: Must the enclosing and the enclosed share an axis of symmetry?

From the diagrams above, one can observe that among the minimally enclosing regular polygons and the regular polygons being enclosed, there is at least one line of symmetry shared among them. Dilworth & Mane [1] showed how it is true when one of the polygons have 3 or 4 vertices, or when the number of sides of the two regular polygons are not coprime, then the number of axes of symmetry is the $\gcd$ of the two numbers of sides. Thus, we hypothesize that there must be at least one common axis of symmetry between the two regular polygons, where one minimally encloses the other, and we anticipate further study on symmetry relations in the future, which may contribute to fully solving the problem on minimal regular polygons enclosing other regular polygons.

References

[1] S. J. Dilworth & S. R. Mane. "On a problem of Croft on optimally nested regular polygons". In: *Journal of Geometry* 99.4 (2010), pp. 43–66. URL: https://link.springer.com/article/10.1007/s00022-011-0065-3.

[2] D. Dureisseix. "Folding optimal polygons from squares". In: *Mathematics Magazine* 79.4 (2006), pp. 272–280. DOI: https://doi.org/10.2307/27642951.

[3] J. E. Wetzel. "Fits and covers." In: *Mathematics Magazine* 76.5 (2003), pp. 349–363. URL: https://www.jstor.org/stable/3654881.