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數學專題習作比賽 (2024/25)
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專題習作題目</td><td colspan="4">Exploring Integer and Heronian Triangles</td></tr>
<tr><td>Name of School</td><td colspan="4">Diocesan Girls’ School</td></tr>
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Exploring Integer and Heronian Triangles

Summary
In the project, we explored some basic properties of integer triangles and Heronian triangles. In part 1 and 2, we explored the general form of integer triangles given an angle and generated the lists of integer and Heronian triangles with special angles with the help of programming. In part 3 and 4, we investigated the combinations of putting two integer triangles together into a larger integer triangle or a cyclic quadrilateral.

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Introduction

0.1 Pythagorean triples

The infamous Pythagorus theorem states the sum of the squares of the two legs of a right-angled triangle (a,b) equals to the square of the hypotenuse(c): $a^2 + b^2 = c^2$. Trios of numbers that can satisfy this equation are called Pythagorean Triples. The Pythagorean triples a,b,c can be generated with the general formula:

$a = m^2 - n^2$

$b = 2mn$

$c = m^2 + n^2$

This raises the question whether there are integer Pythagorean Triples, in which there is: $3^2 + 4^2 = 5^2$, with the two legs of the right-angled triangle as 3 and 4, and the hypotenuse as 5. Triangles with integer sides are termed "integer triangles".

0.2 Cosine formula

Pythagoras theorem is derived from the Cosine theorem, which investigates the relationship between 3 sides in any triangle: $c^2 = a^2 + b^2 - 2ab \cdot cos(\theta^{\circ})$ (where θ° is the angle opposite to side c); In the case of a right-angled triangle, θ° is 90°, and therefore cos(90°) is 0.
Another theorem restraining the relationship between the 3 sides is the triangle inequality: each side of a triangle is smaller than the sum of the other two sides, $a < b + c$; $b < a + c$; $c < a + b$.
Adhering to the Cosine theorem and Triangle inequality, integer triangles with angle degrees other than 90° can be created.

0.3 Integer triangles & Cosine formula rationality

According to Cosine theorem: if $cos\ (C)$ is irrational, then $2ab \cdot cos(C)$, for the product of rational numbers and irrational ones generates an irrational number. $a^2 + b^2$ is rational, for the sum and product of rational numbers are rational. Therefore $a^2 + b^2 - 2ab \cdot cos(C)$ is irrational as well, for the sum of rational and irrational numbers is irrational. Therefore c^2 is irrational, which does not align with integer triangles. Thus the prior condition is that $cos\ (C)$ is rational.

Conclusion

6.1 Summary

Part 1. Deriving integer triangles sides and relationship

$$a=2klq-l^2p,\ b=k^2q-l^2q,\ c=l^2$$

With $m=\frac{k}{l}$ and $r=\frac{p}{q}$ as a rational number. Possible $cos\theta$ include 60°, 90° and 120°.

Part 2. Heronian triangles:

Given the constraints:

1. $mn > k^2$
2. $m \geq n \geq 1$
3. $GCD(m,n,k) = 1$

The sides of a heronian triangle can be parametrized into:

$a = n(m^2+k^2)\ \ b = m(n^2+k^2)\ \ c = (m+n)(mn-k^2)$

The semi-perimeter:

$$s = mn(m+n)$$

The area:

$$A = de(d+f)(df-h^2)$$

Derivation of Heron's formula:

$$\sqrt{s(s+a)(s+b)(s+c)}$$

Heronian triangle angle bisectors:

$d_a = \frac{\sqrt{s(s-2a)}\cdot\sqrt{bc}}{b+c}$ (angle bisector of angle A)

$d_b = \frac{2\sqrt{s(s-b)}\cdot\sqrt{ac}}{a+c}$ (angle bisector of angle B)

$d_c = \frac{2\sqrt{s(s-c)}\cdot\sqrt{ab}}{a+b}$ (angle bisector of angle C)

$d_a\cdot d_b\cdot d_c = \frac{8s\cdot Area\cdot a\cdot b\cdot c}{(a+b)(a+c)(b+c)}$ (Product of 3 angle bisectors)

Inradius:

$r = n-1$ (heronian triangle with a 90° angle)

$r = \frac{(n^2+k^2)(m^2+k^2)}{2(m+n)}$ (heronian triangle with no 90° angle)

Exradius (heronian triangle with a 90° angle):

$$r_a = n+1$$
$$r_b = \frac{n}{n-1}$$
$$r_c = \frac{n}{n+1}$$

Exradius (heronian triangle with no 90° angle):

$$r_a = \frac{m(n^2+k^2)(m^2+k^2)}{2(mn-k^2)}$$

$$r_b = \frac{n(n^2 + k^2)(m^2 + k^2)}{2(mn - k^2)}$$
$$r_c = \frac{nm(n^2 + k^2)(m^2 + k^2)}{2k^2(n + m)}$$

Part 3. Combining integer triangles into larger integer triangles

Case 1 (90°):

- combine two congruent triangles each with a 90° angle to form an isosceles triangle
- combine two similar triangles each with a 90° angle with two separate sides that have the same length to form a larger right-angled triangle
- combine two non-congruent triangles with one side equal to one of the other, so the two right angles are adjacent to form a straight line, a side of the larger triangle formed

Case 2 (60◦ and 120◦):

- One equilateral triangle and one triangle with an angle 120° form a larger triangle if it satisfies that the side c opposite the 120° angle is not the common side
- A scalene 60° triangle and a 120° triangle form a non-equilateral triangle through combining two sides a or b from both triangles
- Two scalene 60° triangle form an equilateral triangle through combining the two c sides, and the two triangles have two other sides of equal length

Case 3 (irrational theta, rational cosine theta):

- infinite number of combinations

Part 4. Combining integer triangles into cyclic quadrilateral

Case 1 (90°):

- When corresponding sides are opposite each other, the two triangles form a rectangle.
- When corresponding sides are adjacent to each other, the two triangles form a kite.
- When the hypotenuses of the two right-angled triangles are the common sides, a cyclic quadrilateral is formed.

Case 2 (60◦ and 120◦):

- When one equilateral triangle and one isosceles triangle form a kite: There are no isosceles triangles with three integer sides
- When one equilateral triangle and one scalene triangle form a general quadrilateral: Only one combination of a 120◦ triangle is needed
- When two scalene 60◦ and 120◦ form a general quadrilateral or a trapezium: The common side needs to be the side c, which is opposite to the angle 60◦/120◦

Case 3 (irrational theta, rational cosine theta):

- there are infinitely many combinations

6.2 Student reflection:

Through this mathematics project on integer triangles, we developed a deeper understanding of trigonometry. Collaborating as a team allowed us to share insights and approaches, enriching our collective understanding of the concepts. This experience has solidified our foundation in geometry and sparked a passion for further exploring mathematical theories and their real-world applications, particularly in areas like geometric modeling.

6.3 Limitation:

- Investigating Heronian triangles was a challenge for us because there are different constraints and parametrizations that we have to adhere to while carefully paying attention to details.
- We did not include non-cyclic quadrilaterals in our project because that investigation would complicate our analysis as it no longer has the properties of cyclic quadrilaterals, such as the requirement for opposite angles to be supplementary. Focusing solely on cyclic shapes allows us to simplify our exploration of integer triangles and make it easier to draw conclusions.

6.4 Future progress:

We will investigate more into construction of different polygons with integer triangles, special lines and centers of heronian triangles. We may also extend into modelling 3D integer pyramids, or into other mathematical topics such as lattice points or elliptic curves with integer lengths etc…

Reference & Appendix

[1]https://structures.uni-heidelberg.de/blog/posts/triangles_integer/index.php

[2]http://tomlr.free.fr/Math%E9matiques/Fichiers%20Claude/Nombres/Carmichael%20-%20diophantine%20analysis%20AAAA.pdf

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