Extract

Brouwer's Fixed Point Theorem: Finding a Fixed Point in Triangles and Extension to Quadrilaterals

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Brouwer's Fixed Point Theorem states that for any function there is at least one point x such that f(x) = x. Such x is called a fixed point of the function. In geometry, the fixed point theorem states that for any two similar figures, in which the small one is placed inside the larger one, at least one fixed point exists. Our team placed $\triangle A'B'C'$ inside $\triangle ABC$, where the two triangles are similar and the vertices and edges of $\triangle A'B'C'$ either touch or lie inside $\triangle ABC$, in hope to find a fixed point P.

Based on knowledge of similar triangles and cyclic quadrilaterals, it was proved that P is the point of intersection of three circumcircles constructed on the sides of the triangles, where each of the circles passes through one pair of corresponding vertices, and the point of intersection between two extended sides of the triangle. P was verified by checking whether the distance ratio between the fixed point and the corresponding vertices was equal. The method was extended to parallelograms and trapeziums, and special cases concerning the touching of corresponding vertices, parallel edges, or overlapping edges were also investigated individually.

Our team also attached Geogebra files in the appendix to illustrate how the method of locating the fixed point works, and how the fixed point always lies on all three circles.

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1. Introduction

Brouwer's Fixed Point Theorem states that for any function there is at least one point x such that $f(x) = x$. Such x is called a fixed point of the function. This theorem can be understood with a very practical geometrical example. For example, if a map of Hong Kong is placed on the ground in Hong Kong, there will always be a point P which makes the position of P on the map the same as that on the real Hong Kong land, which is the 'fixed point'.

The objective of our project is hence formulated in mathematical terms. Consider $\triangle ABC$ and $\triangle A'B'C'$ where $\triangle A'B'C' \sim \triangle ABC$. $\triangle ABC$ is placed on and inside $\triangle A'B'C'$ such that the vertices and sides of $\triangle A'B'C'$ touch or lie inside $\triangle ABC$. Our team attempts to derive a method in finding the fixed point P of $\triangle A'B'C'$ and $\triangle ABC$, extend it to quadrilaterals and a method for verification.

5. Summary and Conclusions

In geometrical terms, Brouwer's Fixed Point Theorem states that for any two similar figures, in which the small one is placed inside the larger one, at least one fixed point exists. In our paper, our team considered a pair of similar triangles, $\triangle ABC$ and $\triangle A'B'C'$, where $\triangle A'B'C'$ is placed inside $\triangle ABC$ such that the vertices and edges of $\triangle A'B'C'$ either touch or lie inside $\triangle ABC$. Our team derived a variety of methods in finding the fixed point P, as elaborated below.

Firstly, our team used basic knowledge of similar triangles and properties of cyclic quadrilaterals to prove that P lies on three constructed circumcircles based on sides of the triangles, where each passes through one pair of corresponding vertices and the point of intersection of one pair of extended sides of the triangles. The distance ratio between the fixed point and the corresponding vertices are equal, implying that it can be presented as an equation relating the distances of P to the vertices, $PA: PA' = PB: PB' = PC: PC'$.

Secondly, our team continued investigating the fixed point of special quadrilaterals. Considering a pair of similar parallelograms, $EFGH$ and $ABCD$, a similar approach of extending sides and constructing the circumcircles is used. The intersection of the four circles is P, verified by the ratio equation $EP: AP = FP: BP = GP: CP = HP: DP$. Although a pair of similar parallelograms was constructed at first for the simplicity of graphic presentation, it can be said that this method also works for trapeziums, as only one pair of parallel lines was considered in the derivation process.

Moreover, the fixed point of circles was investigated. Two triangles were constructed, one inscribing the larger circle, and the other one being inscribed by the smaller circle. A fixed point of the two triangles were found. As the fixed point must lie within both circles, we conclude that the fixed point found is a fixed point of the two circles.

Furthermore, our team recognised the issue that the basic method may not work on some special cases. One example could be when the corresponding sides of the triangles are parallel, as there would be no intersection point between the extended edges, resulting in the inability to construct circumcircles. Our team has investigated different ways of finding fixed points in these special cases. Most of the results showed that the basic method can be generalized and applied directly, while new methods were derived for the cases when the basic method did not work.

Last but not least, our team investigated the case when the smaller triangle is mirrored. The solution derived from our team is that the smaller triangle can be reflected across the fixed point so that the triangles are in the correct orientation of which the fixed point can be solved with the basic method. This approach makes sense as when reflected across the fixed point, all distances from the vertices and the fixed point are preserved, therefore implying that the position of the fixed point would not change.

Based on our findings, future investigation can be carried out on finding the fixed point of similar polygons such as quadrilaterals with no parallel sides. In terms of pure mathematics where polygons and circles do not have a definite orientation, we hypothesize that the number of fixed points is related to whether the figure has rotational symmetry. We sincerely hope that our paper was enjoyable to read. Thank you very much!