The mysterious area of the CIRCLE

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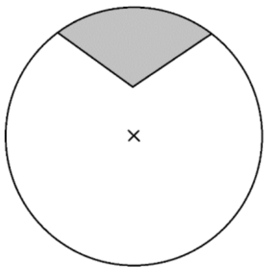
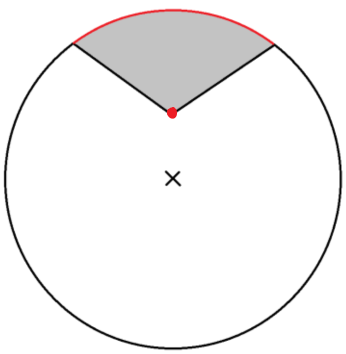
Introduction

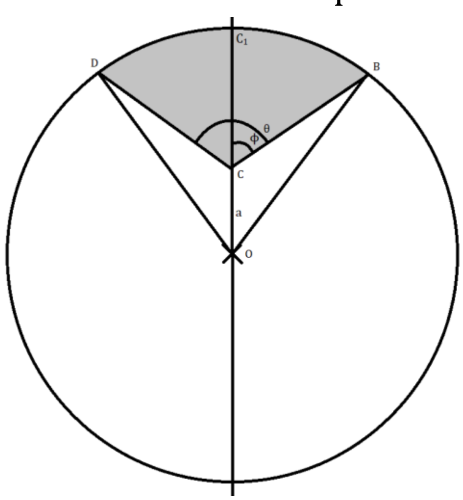
In Math lessons, we have learnt to find the area of a part of a circle, such as a semi-circle, quadrant, sector and segment. What if you were required to find the area of a shape like this?



In this project, we will investigate and create a generalized formula to find the required area of the shape above. This shape is formed with an arc, with the arms connected to an arbitrary point in the circle.

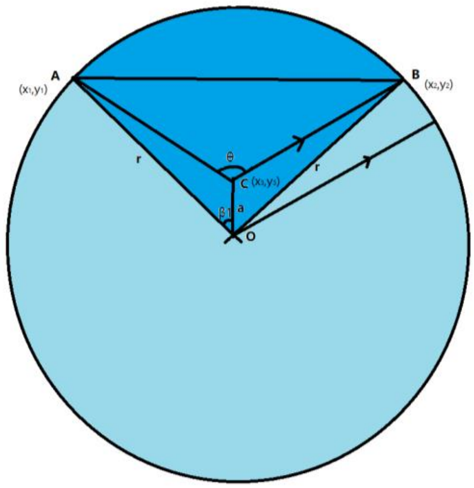


We tried to solve the problem in two different approaches:



Dissecting and fitting method (6 cases):

Cutting the figures into sectors and triangles.



Coordinates geometry (4 cases):

With the 3 pairs of coordinates found, find the area of the shape by shoelace formula.

The given information includes radius (r), α, angle ϕ and angle θ. The definition and reason for the information to be given is that:

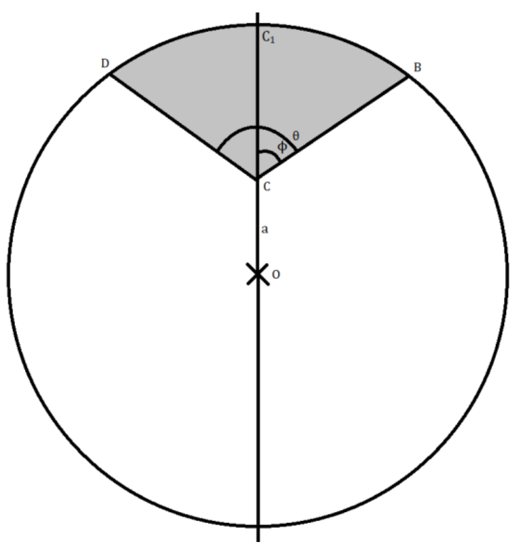
For Method 1:

Radius (r): To limit the size of the circle, which means to limit the size of the area of the shape.

a: It is the distance of the arbitrary point (C) and the centre (O). The reason to give this information is that it can locate the arbitrary point.

Angle ϕ: To locate the position of B.

Angle θ: To locate position of D.



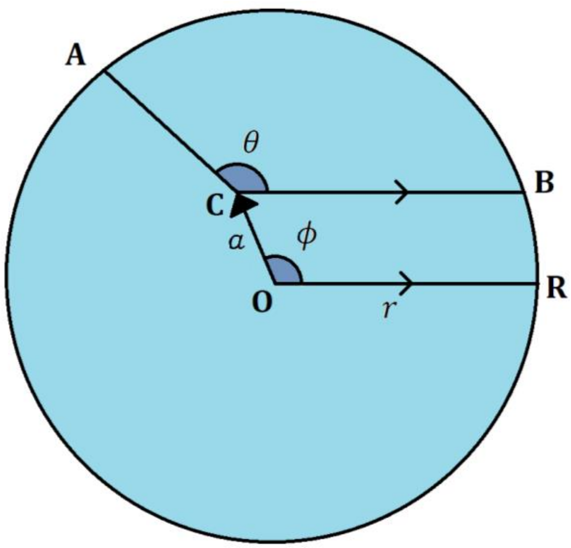
For Method 2:

Radius (r): To limit the size of the circle, which means to limit the size of the area of the shape.

a: It is the distance of the arbitrary point (C) and the centre (O). The reason to give this information is that it can help locate the arbitrary point.

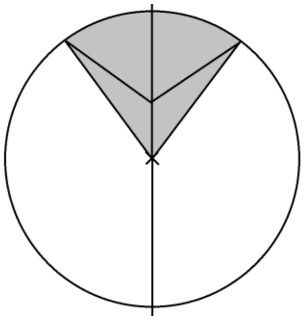
Angle ϕ: To further locate the position of C (so that the position of B can also be located)

Angle θ: To locate position of A.

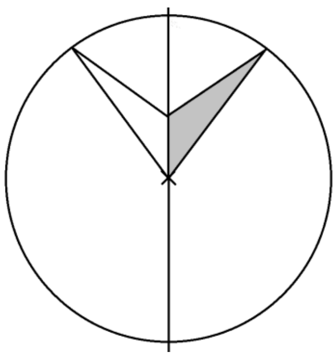


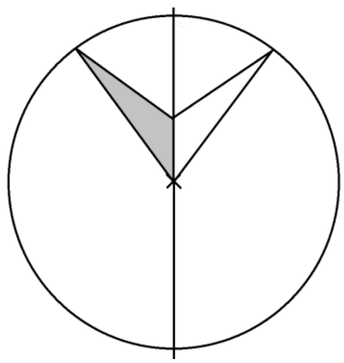
Summary of this method

For this method, we will find the area by dividing it into three components:



One sector and…





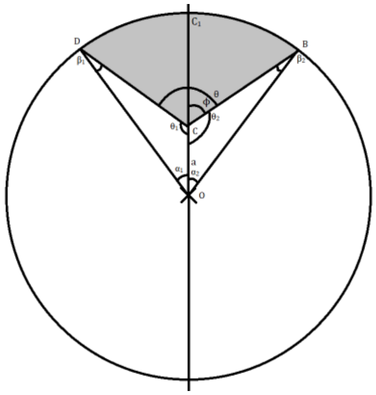
Two triangles

After finding these three areas, we will do addition and subtraction to find out the size of the area.

We generalised all the possibilities into six cases. Noted that the axis of the circle is drawn by extending the line connected to the centre of the circle and the arbitrary point.

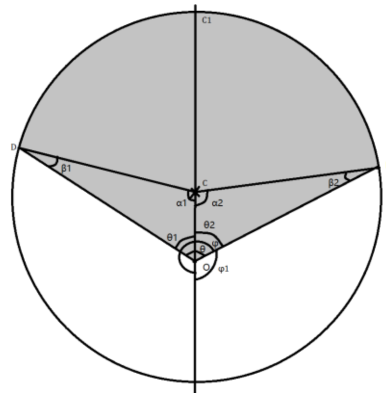
If all six cases can be generalised into the same formula, we can find out the general formula used to calculate the size of the area.

The cases are shown below: (Some cases look reflectionally symmetrical. However, based on the definition we will discuss later, the labelling of angles and points are quite different. Therefore, we separate them into different cases)



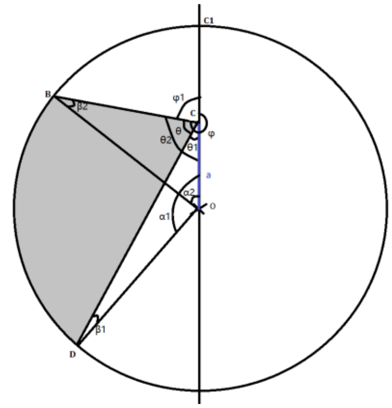
Case 1:

One point of the arc is on the left of the axis, one point is on the right of the axis. The area does not cover the centre



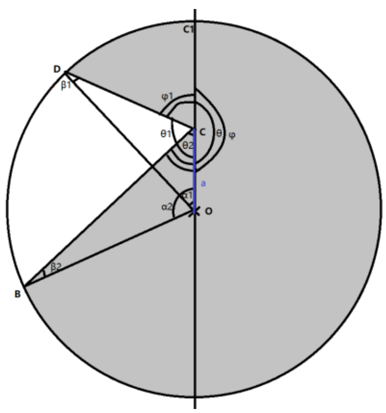
Case 2:

One point of the arc is on the left of the axis, one point is on the right of the axis. The area covers the centre



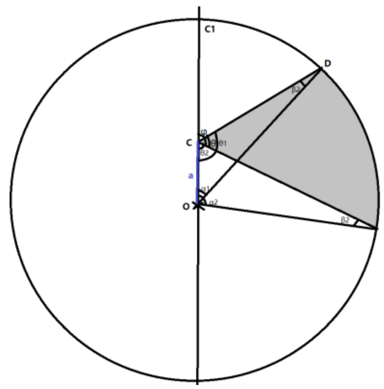
Case 3:

Two points of the arc are on the left of the axis. The area does not cover the centre



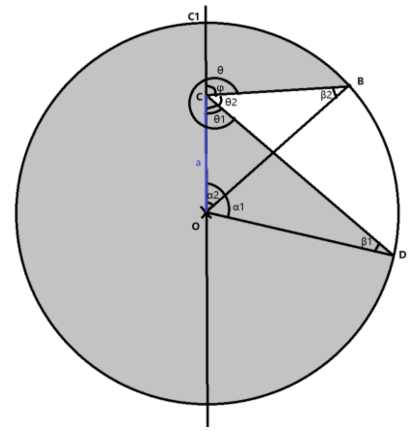
Case 4:

Two points of the arc are on the left of the axis. The area covers the centre



Case 5:

Two points of the arc are on the right of the axis. The area does not cover the centre



Case 6:

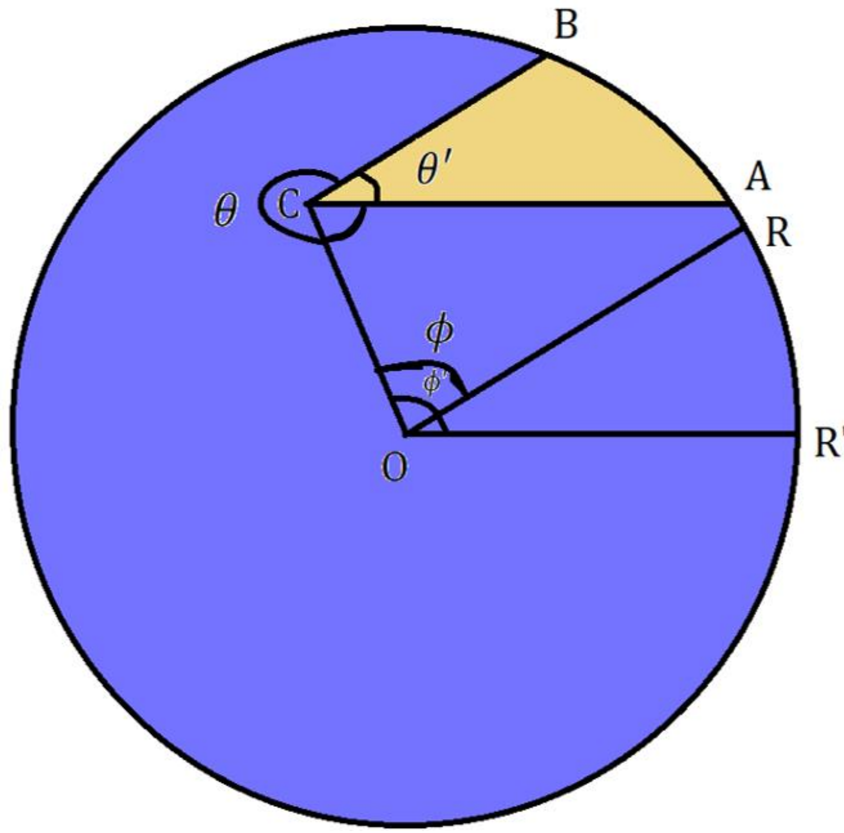
Two points of the arc are on the right of the axis. The area covers the centre

Conclusion

∵Case 1's formula = Case 2's formula = Case 3's formula = Case 4's formula= Case 5's formula= Case 6's formula

∴We can generalize the formula as:

$$\frac{\theta-\sin^{-1}\frac{\mathrm{a}\sin(\theta-\phi)}{r}-\sin^{-1}\frac{\mathrm{a}\sin\phi}{r}}{360^{\circ}}\times\pi r^{2}-\frac{r\,\mathrm{a}\left[\sin\left(\theta-\ \phi-\sin^{-1}\frac{\mathrm{a}\sin(\theta-\phi)}{r}\right)+\sin\left(\phi-\sin^{-1}\frac{\mathrm{a}\sin\phi}{r}\right)\right]}{2}$$



Conclusion:

If we want to find the area of the mysterious area, here is a formula for it:

$$\mathbf{A}_{mysterious} = \mathbf{A}_{\Delta \boldsymbol{ABC}} + (\mathbf{A}_{sector\ \boldsymbol{OAB}} \pm \mathbf{A}_{\Delta \boldsymbol{OAB}})$$

$$\mathbf{A}_{mysterious} = \frac{1}{2}[(x_B y_A - x_A y_B) - (x_C y_A - x_A y_C) + (x_C y_B - x_B y_C)] + \frac{1}{2}[(x_B y_O - x_O y_B) - (x_A y_O - x_O y_A) + (x_A y_B - x_B y_A)] + \mathbf{A}_{sector\ \boldsymbol{OAB}}$$

$$\mathbf{A}_{mysterious} = \frac{1}{2}[(x_B y_A - x_A y_B) - (x_C y_A - x_A y_C) + (x_C y_B - x_B y_C) + (x_A y_B - x_B y_A)] + \mathbf{A}_{sector\ \boldsymbol{OAB}}$$

$$\mathbf{A}_{mysterious} = \frac{1}{2}[(x_C y_B - x_B y_C) - (x_C y_A - x_A y_C)] + \mathbf{A}_{sector\ \boldsymbol{OAB}}$$

$$\mathbf{A}_{mysterious} = \frac{1}{2}\left[\left(a\cos\phi \cdot a\sin\phi - \sqrt{r^2 - a^2\sin^2\phi} \cdot a\sin\phi\right) - \left(a\cos\phi \cdot r\sin[\theta + sin^{-1}\left(\frac{a\sin(\phi-\theta)}{r}\right)] - r\cos\ [\theta + sin^{-1}\left(\frac{a\sin(\phi-\theta)}{r}\right)] \cdot\ a\sin\phi\right)\right] + \mathbf{A}_{sector\ \boldsymbol{OAB}}$$

$$\mathbf{A}_{mysterious} = \frac{1}{2}\left[\left(a\cos\phi \cdot a\sin\phi - \sqrt{r^2 - a^2\sin^2\phi} \cdot a\sin\phi\right) - \left(a\cos\phi \cdot r\sin[\theta + sin^{-1}\left(\frac{a\sin(\phi-\theta)}{r}\right)] - r\cos\ [\theta + sin^{-1}\left(\frac{a\sin(\phi-\theta)}{r}\right)] \cdot\ a\sin\phi\right) + r^2[\theta + \sin^{-1}\left(\frac{a\sin(\phi-\theta)}{r}\right) + \sin^{-1}\left(\frac{a\sin(\phi)}{r}\right)]\right]$$

Reflections and feelings

This is the first time for our team to participate in this competition. During this process to solve the problem we thought of, we have surely learnt a lot.

Since our group is working on a formula, we have to explore all the possibilities and test different cases out with our formula. We also have to find out and state the limitations of our formula. All these challenges and experiences taught us to look at things in different perspectives. For example, a geometric question can be solved using different approaches, given the limited amount of information. Maybe we won't succeed at the first attempt, but we have to try and modify our method to solve the question.

Another thing that we have learnt is that we have to fully understand the nature of the problem we are solving. Sometimes when we look at a mathematical problem, we might have the answer in our mind. However, the most important thing in solving a problem is not the answer, but the process. To obtain the correct method to solve the problem, we need to observe and understand the nature of the problem.

Acknowledgement

Our team would like to express our deepest gratitude to Ms Chan, our teacher-in-charge and other teachers who gave advice about how we may improve our project. But for their help, it would be hard for us to finish this project.