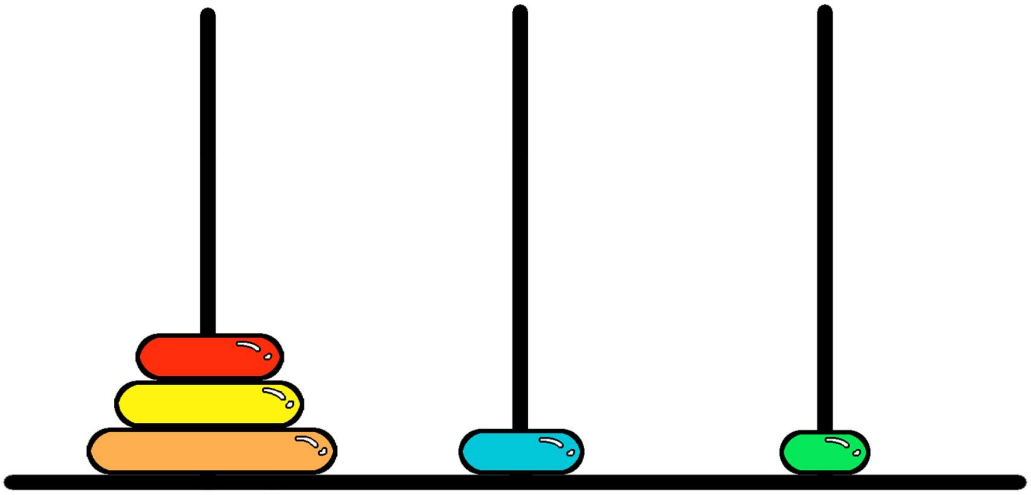
Mathematics Project

Investigation on Least Number of Steps of Tower of Hanoi and its Variations



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1. Introduction

The Tower of Hanoi is a mathematical game invented by the French mathematician Édouard Lucas in 1883. The game consists of 3 rods and a number of disks of different sizes. The original game starts with all the disks on the 1st rod, in ascending order of sizes. The objective is to move all the disks to the 3rd rod, while obeying the following rules:

1. Only one disk can be moved at a time.
2. Each move consists of taking the top disk of one rod and placing it on top of another rod.
3. No larger disk may be placed on top of a smaller disk.

One day, this game was discussed among our classmates. We are curious on what is the least number of steps to solve the Tower of Hanoi. We also found a variation of Tower of Hanoi on the internet, which is One-Step Tower of Hanoi, which added a rule.

Later, our teacher gave us another variation, the Double Tower of Hanoi. We think that this problem is very interesting, so we also decided to find the least number of steps to solve the puzzle.

After finding the formulae in both variations, we started thinking to combine two variations, since one changes the rules, the other one changes the starting position.

In this project, we aim to find out the formulae of least number of steps to solve the Tower of Hanoi, and its variations.

5. Conclusion

After investigating the least number of steps of the original tower of Hanoi and its variations (One-Step, Double, and One-Step Double), we found out the general formula of the least number of steps of the original tower of Hanoi, and its variations. We used the recursive method to find the formula in all problems.

As an extra discovery, we also find out the general formula of the least number of steps with any valid configuration of Tower of Hanoi.

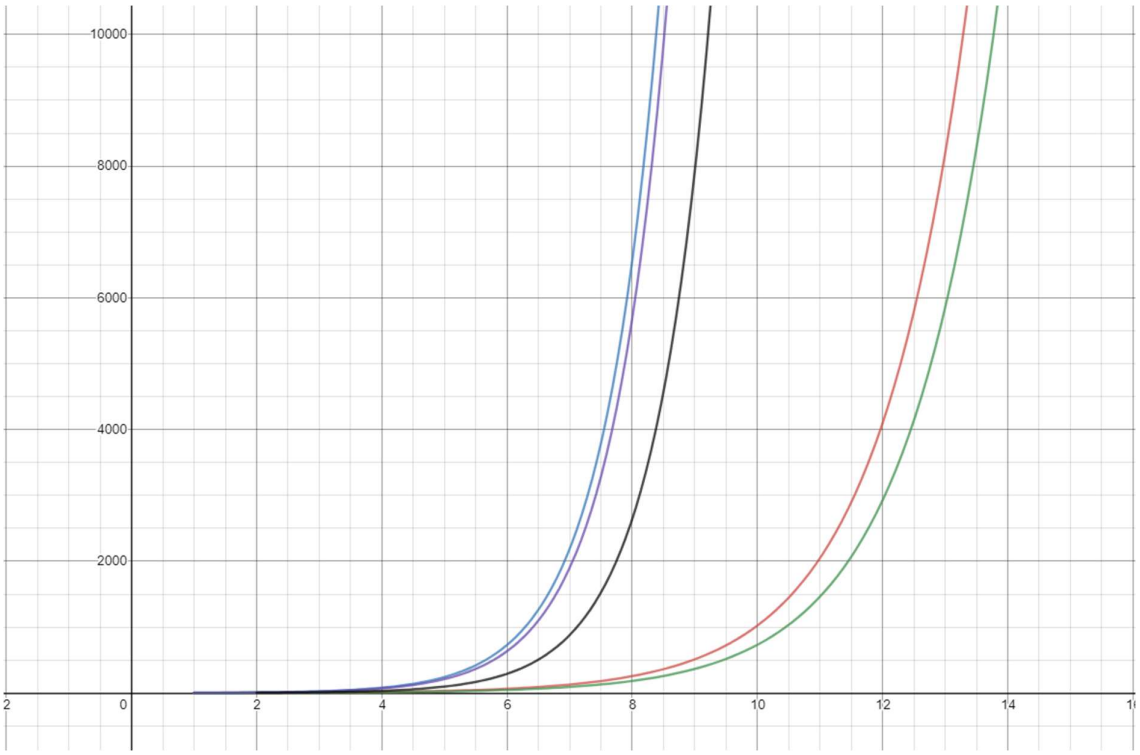

Figure 5a - The least number of steps to solve the Tower of Hanoi and its variations

The x-axis represents n, and the y-axis represents S_n.

Figure 5b - The least number of steps to solve the Tower of Hanoi and its variations (in logarithmic scale, the x-axis represents n, the y-axis represents $\ln(S_n)$)

Red line: The original Tower of Hanoi

Blue line: One-Step Tower of Hanoi

Green line: Double Tower of Hanoi

Purple line: One-Step Double Tower of Hanoi, odd

Black line: One-Step Double Tower of Hanoi, even

Referring to Figure 5a, we can observe that S_n is increasing exponentially.

Referring to Figure 5b, the least number of steps in the One-Step variation increased much faster than the normal and the Double variation. The lines of the One-Step variation are parallel to each other, and the lines of the Double variation and normal are parallel. We can conclude that:

The growth factor of A = B = C > D = E, where A, B, C, D, E is the line of the One-Step Double Tower of Hanoi (odd), One-Step Double Tower of Hanoi (even), One-Step Tower of Hanoi, original Tower of Hanoi, Double Tower of Hanoi respectively.

6. Reference

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3. Desmos - used for drawing graphs
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